

FINITE W-ALGEBRAS

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ABSTRACT. A finite W-algebra is an associative algebra constructed from a semisimple Lie algebra and its nilpotent element. In this survey we review recent developments in the representation theory of W-algebras. We emphasize various interactions between W-algebras and universal enveloping algebras.

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1. INTRODUCTION

Our base field \mathbb{K} is supposed to be algebraically closed and of characteristic zero.

A finite W-algebra is an associative algebra constructed from a pair (\mathfrak{g}, e) , where \mathfrak{g} is a finite dimensional semisimple Lie algebra, and e is a nilpotent element of \mathfrak{g} . A W-algebra should be thought as a generalization of the universal enveloping algebra $U(\mathfrak{g})$. The latter can be considered as the W-algebra for the pair $(\mathfrak{g}, 0)$.

The study of W-algebras traces back to the celebrated paper [32] of Kostant. This paper essentially treats the case when the element e is principal (i.e., the adjoint orbit of e is dense in the nilpotent cone of \mathfrak{g}). Kostant's motivation came basically from the study of Whittaker vectors and of Whittaker models. [32] was followed by the thesis [42] of Lynch who was a student of Kostant. In [42] Kostant's results were (partially) generalized to arbitrary *even* nilpotent elements. During the 80's Whittaker models (in the sense different from Kostant's) were also considered in [46],[47], where they were applied to the study of certain primitive ideals in $U(\mathfrak{g})$.

In the 90's finite W-algebras attracted some attention from mathematical physicists, see, for example, [6],[56],[59]. One of the main motivations for their interest was a relationship between finite and *affine* W-algebras. The latter are certain vertex algebras modeling the so called W-symmetry from Conformal field theory.

In [50] Premet gave a general definition of a W-algebra. Premet's interest to the subject was motivated by the study of non-restricted representations of semisimple Lie algebras in positive characteristic. The paper [50] initiated a lot of work on different, mostly representation theoretic, aspects of W-algebras.

Apart from being of independent interest, finite W-algebras have several connections to other objects studied in Representation theory. Let us summarize these connections.

A) It seems that the most straightforward connection is to the universal enveloping algebras of semisimple Lie algebras. This connection can be informally explained as follows. According to the Orbit method, to an infinite dimensional representation of \mathfrak{g} one should be able to assign a nilpotent orbit in $\mathfrak{g}^*(\cong \mathfrak{g})$. For instance, to a "nice" (e.g., irreducible) Harish-Chandra \mathfrak{g} -bimodule one assigns a dense orbit in its associated variety. Then there is a hope (that sometimes converts into proofs) that one can reduce the study of an infinite dimensional \mathfrak{g} -module to the study of a *finite dimensional* module over the W-algebra corresponding to the nilpotent orbit in interest. A relationship between the W-algebras and $U(\mathfrak{g})$ is studied, for example, in [10],[26],[37]-[40],[51]-[53],[57].

B) There is a connection between the representation theory of W-algebras (in characteristic zero) and that of semisimple Lie algebras in positive characteristic. In a sentence, any reduced enveloping algebra turns out to be Morita equivalent to an appropriate reduced W-algebra. One can relate the representation theories of W-algebras in positive and in zero characteristics. This relationship was successfully used in Premet's papers, see [49],[50],[52],[53].

C) For classical Lie algebras there is a connection between W-algebras and (twisted) Yangians. This connection was first discovered in [56] and then studied further in [7],[8],[12],[13],[55].

Also W-algebras are related to the (cyclotomic quotients of) degenerate affine Hecke algebras, [14].

D) As we mentioned above, finite W-algebras are related to their affine counterparts. This relation can be made formal. To any vertex algebra one can assign an associative algebra called the *Zhu algebra*. The importance of the Zhu algebra is that its representation theory controls much of the representation theory of the initial vertex algebra. A finite W-algebra is closely related to the Zhu algebra of the corresponding affine W-algebra. For details the reader is referred to [17].

In the present paper we are mostly interested in A). We also briefly explain C), while B) remains almost untouched and we do not discuss D) at all. Therefore we suppress the adjective "finite" while speaking about W-algebras. Another review on W-algebras [60] by W. Wang have already appeared. Some topics not discussed (or discussed very briefly) in our survey can be found there.

This paper is organized as follows. In Section 2 we discuss topics related to a definition of a W-algebra via Hamiltonian reduction, which is essentially due to Premet. In Section 3 we explain the definition of a W-algebra based on the Deformation quantization, [37]. The next three sections describe connections between W-algebras and $U(\mathfrak{g})$. In Section 4 we discuss category equivalences between certain categories of modules for W-algebras and for $U(\mathfrak{g})$. Section 5 describes a relationship between the sets of two-sided ideals in the two algebras. This description leads to a (partial) classification of irreducible finite dimensional representations of W-algebras in terms of primitive ideals of $U(\mathfrak{g})$. Section 6 deals with one-dimensional modules over W-algebras. Finally in Section 7 we explain the connection C) above mostly for \mathfrak{g} of type A.

In the beginning of each section its content is described in more detail.

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Notation and conventions. Throughout the paper G is a connected reductive group, \mathfrak{g} is its Lie algebra. We choose a nilpotent element $e \in \mathfrak{g}$ and pick $h, f \in \mathfrak{g}$ forming an \mathfrak{sl}_2 -triple with e , i.e., $[h, e] = 2e, [h, f] = -2f, [e, f] = h$. Let \mathbb{O} denote the G -orbit of e . Also we fix a G -invariant non-degenerate symmetric form (\cdot, \cdot) on \mathfrak{g} . Using this form, we identify \mathfrak{g} with \mathfrak{g}^* .

We write \mathcal{U} for the universal enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} . By \mathcal{Z} we denote the center of \mathcal{U} . This is a polynomial algebra.

Let us also list some standard notation used below.

A^{op}	the opposite algebra of an algebra A .
$\text{Ann}_A(M)$	the annihilator of an A -module M .
$\text{End}_A(M)$	the algebra of endomorphisms of an A -module M .
$\text{End}(M)$	$:= \text{End}_{\mathbb{K}}(M)$.
$\text{gr } V$	the associated graded vector space of a filtered vector space V .
H°	the unit component of an algebraic group H .
$\mathbb{K}[X]$	the algebra of regular functions on a variety X .
$\mathbb{K}[X]_Y^\wedge$	the algebra of functions on the completion of a variety X along a subvariety Y .
T^*X	the cotangent bundle of a smooth variety X .

$V(\mathcal{M})$	the associated variety of a finitely generated \mathcal{U} -module \mathcal{M} .
$\mathfrak{z}(\mathfrak{h})$	the center of a Lie algebra \mathfrak{h} .
$\mathfrak{z}_{\mathfrak{h}}(\mathfrak{f})$	the centralizer of \mathfrak{f} in a Lie algebra \mathfrak{h} .

2. W-ALGEBRAS VIA HAMILTONIAN REDUCTION

In this section we discuss developments leading to and related to Premet's definition of a W-algebra given in [50]. The first such development is, of course, Kostant's work, [32], where the case of a principal nilpotent element was treated. We describe (very few of) Kostant's results in Subsection 2.1. Then in Subsection 2.2 we mention a generalization of Kostant's constructions to the case of an even nilpotent element due to Lynch, [42]. In Subsection 2.3 we provide Premet's definition in the form of a quantum Hamiltonian reduction. In Subsection 2.4 we show that the "quasiclassical limit" of a W-algebra is the algebra of functions on a *Slodowy slice* that is a transverse slice to a nilpotent orbit in \mathfrak{g} introduced in [58]. In Subsection 2.5 we mention several ramifications of Premet's definition and in Subsection 2.6 discuss some properties of W-algebras that can be proved using this definition.

2.1. Kostant's results: the case of a principal nilpotent element. In this subsection we will explain (some of) Kostant's results, Section 2 of [32].

Suppose the nilpotent element e is principal. Set

$$(2.1) \quad \mathfrak{g}(i) := \{\xi \in \mathfrak{g} \mid [h, \xi] = i\xi\}, \mathfrak{p} := \bigoplus_{i \geq 0} \mathfrak{g}(i), \mathfrak{m} := \bigoplus_{i < 0} \mathfrak{g}(i), \chi := (e, \cdot).$$

Let us describe $\mathfrak{p}, \mathfrak{m}, e, \chi$ in more conventional terms. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of \mathfrak{g} , $\Delta \subset \mathfrak{h}^*$ the corresponding root system, and Π a system of simple roots in Δ . Further, for $\alpha \in \Delta$ let e_{α} denote a corresponding weight vector in \mathfrak{g} . Finally, let ρ^{\vee} denote half the sum of all positive coroots (=the sum of all fundamental co-weights). Replacing (e, h, f) with a G -conjugate triple, we may assume that $h = 2\rho^{\vee}$ and $e = \sum_{\alpha \in \Pi} e_{\alpha}$. So \mathfrak{p} becomes the positive Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, \mathfrak{m} becomes the negative maximal nilpotent subalgebra \mathfrak{n}_{-} , while χ is a non-degenerate character of \mathfrak{m} .

Define the shift $\mathfrak{m}_{\chi} := \{\xi - \langle \chi, \xi \rangle, \xi \in \mathfrak{m}\}$ of \mathfrak{m} . Then, thanks to the PBW theorem, we get

$$(2.2) \quad \mathcal{U} = U(\mathfrak{p}) \oplus \mathcal{U}\mathfrak{m}_{\chi}.$$

Using this decomposition, one can define an action of \mathfrak{m} on $U(\mathfrak{p})$. Namely, identify $U(\mathfrak{p})$ with the quotient $\mathcal{U}/\mathcal{U}\mathfrak{m}_{\chi}$ using (2.2). The adjoint action of \mathfrak{m} on \mathcal{U} descends to $\mathcal{U}/\mathcal{U}\mathfrak{m}_{\chi}$. Using the identification $U(\mathfrak{p}) \cong \mathcal{U}/\mathcal{U}\mathfrak{m}_{\chi}$, we get an \mathfrak{m} -action on $U(\mathfrak{p})$.

By definition, a W-algebra $U(\mathfrak{g}, e)$ is the invariant subalgebra $U(\mathfrak{p})^{\mathfrak{m}}$. In other words, $U(\mathfrak{g}, e)$ is the *quantum Hamiltonian reduction*

$$(\mathcal{U}/\mathcal{U}\mathfrak{m}_{\chi})^{\text{ad } \mathfrak{m}} := \{a + \mathcal{U}\mathfrak{m}_{\chi} : [\xi, a] \in \mathcal{U}\mathfrak{m}_{\chi}, \forall \xi \in \mathfrak{m}\}.$$

The multiplication on the last space is defined by $(a + \mathcal{U}\mathfrak{m}_{\chi})(b + \mathcal{U}\mathfrak{m}_{\chi}) = ab + \mathcal{U}\mathfrak{m}_{\chi}$.

It turns out that $U(\mathfrak{g}, e)$ is naturally isomorphic to the center \mathcal{Z} of \mathcal{U} . Namely, the inclusion $\mathcal{Z} \hookrightarrow \mathcal{U}$ gives rise to a natural map $\mathcal{Z} \rightarrow \mathcal{U}/\mathcal{U}\mathfrak{m}_{\chi}$. Its image clearly consists of \mathfrak{m} -invariants. So we get a homomorphism $\mathcal{Z} \rightarrow U(\mathfrak{g}, e)$. By Theorem 2.4.1 in [32], this homomorphism is an isomorphism. In particular, we get an embedding of \mathcal{Z} into $U(\mathfrak{p})$. This embedding is of importance in the quantization of Toda systems, see [33].

2.2. Generalization: the case of even e . Recall that e is called *even* if all eigenvalues of $\text{ad } h$ on \mathfrak{g} are even. Define $\mathfrak{g}(i), \mathfrak{p}, \mathfrak{m}, \chi$ by (2.1). It is clear that \mathfrak{p} is a parabolic subalgebra of \mathfrak{g} and \mathfrak{m} is the nilpotent radical of the opposite parabolic. In [42] Lynch generalized Kostant's definition and introduced an algebra $U(\mathfrak{g}, e) := U(\mathfrak{p})^{\mathfrak{m}} = (\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi)^{\text{ad } \mathfrak{m}}$.

There is an embedding $U(\mathfrak{g}, e) \hookrightarrow U(\mathfrak{g}(0))$ sometimes called the *generalized Miura transform*. It is obtained by restricting the natural projection $U(\mathfrak{p}) \twoheadrightarrow U(\mathfrak{g}(0))$ to $U(\mathfrak{g}, e) \subset U(\mathfrak{p})$. The restriction is injective by [42], Corollary 2.3.2.

2.3. Definition of $U(\mathfrak{g}, e)$: the general case. Now let $e \in \mathfrak{g}$ be an arbitrary (nonzero) nilpotent element. Let the decomposition $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ and the element $\chi \in \mathfrak{g}^*$ be given by (2.1). Following Premet, [50], we still can define a W-algebra $U(\mathfrak{g}, e)$ as the quantum Hamiltonian reduction $(\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi)^{\text{ad } \mathfrak{m}}$ provided we can define a suitable analog of the subalgebra $\mathfrak{m} \subset \mathfrak{g}$ considered in the previous subsection.

A subalgebra \mathfrak{m} we need is constructed as follows. Consider a skew-symmetric form ω_χ on \mathfrak{g} given by $\omega_\chi(\xi, \eta) = \langle \chi, [\xi, \eta] \rangle$. It follows easily from the representation theory of \mathfrak{sl}_2 that the restriction of ω_χ to the subspace $\mathfrak{g}(-1)$ is non-degenerate. Pick a lagrangian subspace $l \subset \mathfrak{g}(-1)$ and set $\mathfrak{m} := l \oplus \bigoplus_{i \leq -2} \mathfrak{g}(i)$.

It is clear that \mathfrak{m} is a subalgebra in \mathfrak{g} consisting of nilpotent elements. Also since ω_χ vanishes on l , we see that $\langle \chi, [\mathfrak{m}, \mathfrak{m}] \rangle = 0$. So χ is indeed a character of \mathfrak{m} .

We set $U(\mathfrak{g}, e) := (\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi)^{\text{ad } \mathfrak{m}}$. The reader should notice that, a priori, this definition is ambiguous: \mathfrak{m} and hence $U(\mathfrak{g}, e)$ depend on the choice of l . However, we will see in Subsection 2.5 that two W-algebras constructed using different choices of l are canonically isomorphic.

We finish the subsection with a few historical remarks. Direct analogs of \mathfrak{m} and of its shift \mathfrak{m}_χ in the setting of finite Chevalley groups first appeared in [30]. Then Moeglin used \mathfrak{m}_χ to define "Whittaker models" for primitive ideals of $U(\mathfrak{g})$, [46], [47]. We will describe her results in more detail in Section 6. Later the subalgebra \mathfrak{m} played an important role in Premet's proof of the Kac-Weisfeller conjecture on the dimension of a non-restricted representation of a semisimple Lie algebra in positive characteristic, see [49].

2.4. Classical counterpart: the Slodowy slice. The algebra $U(\mathfrak{g}, e)$ has an interesting filtration, called the *Kazhdan filtration*.

To define it we first introduce a new filtration on \mathcal{U} . Recall that the algebra \mathcal{U} has the standard, PBW filtration: the subspace $F_i^{\text{st}} \mathcal{U}$ of elements of degree $\leq i$, by definition, is spanned by all monomials $\xi_1 \dots \xi_j, j \leq i, \xi_1, \dots, \xi_j \in \mathfrak{g}$. For $j \in \mathbb{Z}$ set $\mathcal{U}(j) := \{u \in \mathcal{U} | [h, u] = ju\}$. Define the *Kazhdan filtration* $K_i \mathcal{U}$ on \mathcal{U} by $K_i \mathcal{U} := \sum_{2j+k \leq i} F_j^{\text{st}} \mathcal{U} \cap \mathcal{U}(k)$. We remark that the associated graded algebra of \mathcal{U} with respect to the Kazhdan filtration is still naturally isomorphic to the symmetric algebra $S(\mathfrak{g})$.

Being a subquotient of \mathcal{U} , the algebra $U(\mathfrak{g}, e)$ has a Kazhdan filtration $K_i U(\mathfrak{g}, e)$ inherited from \mathcal{U} . We remark that $K_0 \mathcal{U} \subset \mathbb{K} + \mathcal{U}\mathfrak{m}_\chi$ so the Kazhdan filtration on $U(\mathfrak{g}, e)$ is positive in the sense that $K_0 U(\mathfrak{g}, e) = \mathbb{K}$.

It turns out that the associated graded algebra $\text{gr } U(\mathfrak{g}, e)$ of $U(\mathfrak{g}, e)$ is naturally isomorphic to the algebra of functions on the *Slodowy slice* $S := e + \ker \text{ad}(f)$, [58] (in the case when e is principal S appeared in [31]). It follows from the representation theory of \mathfrak{sl}_2 that S is transverse to \mathbb{O} . In the sequel it will be convenient for us to consider S as an affine subspace in \mathfrak{g}^* via the identification $\mathfrak{g} \cong \mathfrak{g}^*$. In particular, $\chi \in S$.

We need an action of the one-dimensional torus \mathbb{K}^\times on \mathfrak{g}^* that stabilizes S and contracts it to χ . Namely, the \mathfrak{sl}_2 -triple (e, h, f) defines a homomorphism $\text{SL}_2(\mathbb{K}) \rightarrow G$. The group \mathbb{K}^\times

is embedded into $\mathrm{SL}_2(\mathbb{K})$ via $t \mapsto \mathrm{diag}(t, t^{-1})$. Composing these two homomorphisms we get a homomorphism (in fact, an embedding) $\gamma : \mathbb{K}^\times \rightarrow G$. For $\xi \in \mathfrak{g}(i)$ we have $\gamma(t) \cdot \xi = t^i \xi$. Consider a \mathbb{K}^\times -action (called the Kazhdan action) on \mathfrak{g}^* given by $t \cdot \alpha = t^{-2} \gamma(t) \alpha$. This action fixes χ . Also it is easy to see that it preserves S . Finally, the representation theory of \mathfrak{sl}_2 implies that $\ker \mathrm{ad}(f) \subset \bigoplus_{i \leq 0} \mathfrak{g}(i)$. It follows that the action of \mathbb{K}^\times contracts S to χ : $\lim_{t \rightarrow \infty} t \cdot s = \chi$ for any $s \in S$. The contraction property has several very nice corollaries. For example, S intersects an adjoint orbit \mathbb{O}' if and only if $\mathbb{O} \subset \overline{\mathbb{K}^\times \mathbb{O}'}$ and in this case the intersection $S \cap \mathbb{O}'$ is transversal.

The Kazhdan action gives rise to a (positive) grading on the algebra $\mathbb{K}[S]$ of regular functions on S . The following result was essentially obtained by Premet, [50], Theorem 4.6 (Kostant and Lynch also proved this in the special cases they considered).

Theorem 2.1. *There is an isomorphism $\mathrm{gr} U(\mathfrak{g}, e) \cong \mathbb{K}[S]$ of graded algebras.*

As was shown by Gan and Ginzburg, [25], this result is a manifestation of the "quantization commutes with reduction" principle, see Subsection 2.5 for details.

2.5. Ramifications. First, let us mention the work of Gan and Ginzburg, [25], where they gave a ramification of Premet's definition showing, in particular, that $U(\mathfrak{g}, e)$ does not depend on the choice of $l \subset \mathfrak{g}(-1)$.

Namely, let $l \subset \mathfrak{g}(-1)$ be an arbitrary isotropic subspace of $\mathfrak{g}(-1)$ (e.g., $\{0\}$). Let l^\perp denote the skew-orthogonal complement to l in $\mathfrak{g}(-1)$. Set $\mathfrak{m}^l := l \oplus \bigoplus_{i \leq -2} \mathfrak{g}(i)$, $\mathfrak{n}^l := l^\perp \oplus \bigoplus_{i \leq -2} \mathfrak{g}(i)$. Then $\mathfrak{m}^l \subset \mathfrak{n}^l$, \mathfrak{n}^l consists of nilpotent elements, and $\langle \chi, [\mathfrak{m}_l, \mathfrak{n}_l] \rangle = 0$. Let N^l be the connected subgroup of G with Lie algebra \mathfrak{n}^l . Then \mathfrak{m}^l and the character $\chi : \mathfrak{m}^l \rightarrow \mathbb{K}$ are stable under the adjoint action of N^l . So N^l acts naturally on the quotient $\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi^l$, where $\mathfrak{m}_\chi^l := \{\xi - \langle \chi, \xi \rangle, \xi \in \mathfrak{m}^l\}$. Let $U(\mathfrak{g}, e)^l := (\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi^l)^{N^l}$ be the space of invariants. It is easy to check that it has a natural algebra structure. It also has a Kazhdan filtration $K_i U(\mathfrak{g}, e)^l$, compare with the previous subsection.

Now let us remark that for $l_1 \subset l_2$ we have a natural \mathcal{U} -module homomorphism $\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi^{l_1} \rightarrow \mathcal{U}/\mathcal{U}\mathfrak{m}_\chi^{l_2}$ that gives rise to a filtered algebra homomorphism $U(\mathfrak{g}, e)^{l_1} \rightarrow U(\mathfrak{g}, e)^{l_2}$. It turns out that the latter is an isomorphism.

Also Gan and Ginzburg gave a very transparent explanation of an isomorphism $\mathrm{gr} U(\mathfrak{g}, e) \cong \mathbb{K}[S]$. Namely, consider the restriction map $\pi : \mathfrak{g}^* \rightarrow \mathfrak{m}^*$. The affine subspace $\pi^{-1}(\chi|_{\mathfrak{m}^l}) \subset \mathfrak{g}^*$ is N^l -stable. Also it is easy to see that $S \subset \pi^{-1}(\chi|_{\mathfrak{m}^l})$. So we can consider a morphism $N^l \times S \rightarrow \pi^{-1}(\chi|_{\mathfrak{m}^l})$, $(n, s) \mapsto ns$. According to [25], this is an isomorphism (of algebraic varieties). Therefore $\mathbb{K}[S]$ gets identified with the classical Hamiltonian reduction $(S(\mathfrak{g})/S(\mathfrak{g})\mathfrak{m}_\chi^l)^{N^l}$.

Another ramification of the original definition of $U(\mathfrak{g}, e)$ comes from the notion of a good grading on \mathfrak{g} , [20]. A grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ is said to be *good* for e if $e \in \mathfrak{g}(2)$ and $\ker \mathrm{ad}(e) \subset \bigoplus_{i \geq 0} \mathfrak{g}(i)$. For instance, the grading given by (2.1) is good. For a comprehensive study of good gradings see [20].

Given a good grading on \mathfrak{g} , one constructs $\mathfrak{m} \subset \mathfrak{g}$ and defines $U(\mathfrak{g}, e)$ using \mathfrak{m} analogously to the above. The algebra $U(\mathfrak{g}, e)$ does not depend on the choice of a good grading. This was first proved in [9].

The definition involving an arbitrary good grading is often useful. For example, one can sometimes find an *even* good grading when e is not even itself and embed $U(\mathfrak{g}, e)$ into $U(\mathfrak{p})$ for an appropriate parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$, compare with Subsection 2.2. This is always the case when $\mathfrak{g} \cong \mathfrak{sl}_n$, see [12], Introduction.

Also it is worth mentioning that there is a related definition of $U(\mathfrak{g}, e)$ via the BRST quantization procedure which was used by physicists in the 90-s, see [6]. The proof that the BRST definition is equivalent to the one given above was obtained in [16]. See also [60], Section 3.

2.6. Additional properties of $U(\mathfrak{g}, e)$. We want to make a few remarks about other properties of $U(\mathfrak{g}, e)$.

Recall that \mathcal{Z} stands for the center of \mathcal{U} . Restricting the natural map $\mathcal{U}^{\text{adm}} \rightarrow U(\mathfrak{g}, e)$ to $\mathcal{Z} \subset \mathcal{U}^{\text{adm}}$, we get an algebra homomorphism $\mathcal{Z} \rightarrow U(\mathfrak{g}, e)$. By [50], 6.2, this homomorphism is an embedding. It is clear that the image of \mathcal{Z} lies in the center of $U(\mathfrak{g}, e)$. Further, according to the footnote to Question 5.1 in [51], the image of \mathcal{Z} actually coincides with the center of $U(\mathfrak{g}, e)$ (Premet attributes the proof to Ginzburg). This generalizes Kostant's result mentioned in Subsection 2.1.

Also we remark that there is a natural action of the group $Q := Z_G(e, h, f)$ on $U(\mathfrak{g}, e)$. Namely, take $l = \{0\}$ in the Gan and Ginzburg definition. Then Q stabilizes both \mathfrak{m}_χ^l and N^l and so acts on $U(\mathfrak{g}, e)^l$. Let \mathfrak{q} stand for the Lie algebra of Q . In [51] Premet constructed a Lie algebra embedding $\mathfrak{q} \hookrightarrow U(\mathfrak{g}, e)$ such that the adjoint action of \mathfrak{q} on $U(\mathfrak{g}, e)$ coincides with the differential of the Q -action.

3. W-ALGEBRAS VIA DEFORMATION QUANTIZATION

In this section we review the definition of W-algebras from [37]. It is based on Deformation quantization: a W-algebra is realized as an algebra of G -invariants in a quantization of a certain affine symplectic G -variety (called an *equivariant Slodowy slice*). In Subsection 3.1 we briefly explain generalities on star-products and on Fedosov's method to construct them. In Subsection 3.2 we present constructions of equivariant Slodowy slices and of W-algebras. Finally, in Subsection 3.3 we present a very important basic result on W-algebras, the *decomposition theorem*.

3.1. Fedosov quantization. In this subsection X is a smooth affine variety equipped with a symplectic form ω . Let $\{\cdot, \cdot\}$ denote the Poisson bracket on $\mathbb{K}[X]$ induced by ω . Let a reductive group \tilde{G} act on X preserving ω . By ξ_X we denote the image of $\xi \in \tilde{\mathfrak{g}}$ under the homomorphism $\tilde{\mathfrak{g}} \rightarrow \text{Der}(\mathbb{K}[X])$ induced by the action.

We suppose that the \tilde{G} -action is Hamiltonian, that is, admits a *moment map* $\mu : X \rightarrow \tilde{\mathfrak{g}}^*$, i.e., a \tilde{G} -equivariant morphism having the following property: for $H_\xi := \mu^*(\xi)$, $\xi \in \tilde{\mathfrak{g}}$, we have $\{H_\xi, \cdot\} = \xi_X$. Finally, we suppose that X is equipped with a \mathbb{K}^\times -action that commutes with \tilde{G} and satisfies $t\omega = t^2\omega$, $t.H_\xi = t^2H_\xi$ for all $t \in \mathbb{K}^\times$, $\xi \in \tilde{\mathfrak{g}}$. We will present examples of this situation below.

By a *star-product* on $\mathbb{K}[X]$ (or on X) we mean a \mathbb{K} -bilinear map $* : \mathbb{K}[X] \times \mathbb{K}[X] \rightarrow \mathbb{K}[X][[\hbar]]$, $(f, g) \mapsto f * g := \sum_{i=0}^{\infty} D_i(f, g)\hbar^{2i}$ satisfying the following axioms:

- (a) The associativity axiom: a natural extension of $*$ to a $\mathbb{K}[[\hbar]]$ -bilinear map $\mathbb{K}[X][[\hbar]] \times \mathbb{K}[X][[\hbar]] \rightarrow \mathbb{K}[X][[\hbar]]$ is an associative product, and $1 \in \mathbb{K}[X] \subset \mathbb{K}[X][[\hbar]]$ is a unit for $*$.
- (b) The compatibility axiom: $D_0(f, g) = fg$, $D_1(f, g) - D_1(g, f) = \{f, g\}$. Equivalently, $f * g \equiv fg \pmod{\hbar^2}$ and $[f, g] \equiv \hbar^2\{f, g\} \pmod{\hbar^4}$.
- (c) The locality axiom: D_i is a bidifferential operator of order at most i (i.e., for any fixed f the map $\mathbb{K}[X] \rightarrow \mathbb{K}[X] : g \mapsto D_i(f, g)$, is a differential operator of order at most i , and the same for any fixed g).

When we consider $\mathbb{K}[X][[\hbar]]$ as an algebra with respect to the star-product, we call it a *quantum algebra*.

We remark that the usual definition of a star-product looks like $f * g = \sum_{i=0}^{\infty} D_i(f, g) \hbar^i$ and in our definition we have \hbar^2 instead of \hbar . The reason for this ramification is that our version is better compatible with the *Rees algebra construction*. This construction allows to pass from filtered \mathbb{K} -algebras to graded $\mathbb{K}[\hbar]$ -algebras.

We will also need $*$ to be compatible with the \tilde{G} - and \mathbb{K}^\times -actions on X .

(d) \tilde{G} -invariance: $D_i : \mathbb{K}[X] \otimes \mathbb{K}[X] \rightarrow \mathbb{K}[X]$ is \tilde{G} -equivariant.

(e) Homogeneity: D_i has degree $-2i$ with respect to \mathbb{K}^\times : i.e., for $f, g \in \mathbb{K}[X]$ of degrees j, k the element $D_i(f, g)$ has degree $k + j - 2i$.

Under the conditions (d) and (e), the product $\tilde{G} \times \mathbb{K}^\times$ acts on $\mathbb{K}[X][[\hbar]]$ by automorphisms with $g.\hbar = \hbar, t.\hbar = t\hbar$ for all $g \in \tilde{G}, t \in \mathbb{K}^\times$.

It turns out that a star-product on X satisfying additionally (d) and (e) always exists. It is provided, for example, by Fedosov's construction, [21],[22]. Fedosov constructed a star-product on a C^∞ -manifold starting from a *symplectic connection* ∇ and a closed $\mathbb{K}[[\hbar^2]]$ -valued form Ω . By definition, a symplectic connection is a torsion-free connection on the tangent bundle such that the symplectic form is flat. Fedosov's construction can be carried over to the algebraic setting as long as a variety in consideration admits a symplectic connection. Since X is affine, this is the case, and, moreover, one can, in addition, assume that a symplectic connection is $\tilde{G} \times \mathbb{K}^\times$ -invariant, see [37], Proposition 2.2.2. For our purposes, it will be enough to consider the original construction from [21], where Ω is not used (i.e., equals 0).

The following proposition follows from results of Fedosov, see [38], Theorem 2.1.2 for details.

Proposition 3.1. *Let X be as above, and ∇ be a $\tilde{G} \times \mathbb{K}^\times$ -invariant symplectic connection on X . Further, let $*$ be the star-product produced from ∇ by the Fedosov construction. Then $*$ is \tilde{G} -invariant and homogeneous. Moreover, the map $\xi \mapsto H_\xi$ is a quantum comoment map for the \tilde{G} -action on $\mathbb{K}[X][[\hbar]]$, i.e., $\frac{1}{\hbar^2}[H_\xi, f] = \xi_X f$ for all $f \in \mathbb{K}[X][[\hbar]], \xi \in \tilde{\mathfrak{g}}$.*

Also, according to Fedosov, $*$ does not depend on the choice of ∇ up to a suitably understood isomorphism, see, for example, [37], Proposition 2.2.5, for details.

Let us consider two standard examples.

The first example is easy. Let V be a vector space equipped with a non-degenerate form $\omega \in \bigwedge^2 V^*$. Let \tilde{G} act on V via a homomorphism $\tilde{G} \rightarrow \mathrm{Sp}(V)$. Pick a homomorphism $\beta : \mathbb{K}^\times \rightarrow \mathrm{Sp}(V)^{\tilde{G}}$ and define a \mathbb{K}^\times -action on V^* by $t.\alpha = t^{-1}\beta(t)\alpha$. So we get a symplectic variety $X = V^*$ equipped with a $\tilde{G} \times \mathbb{K}^\times$ -action satisfying the assumptions above with the moment map given by $\langle \mu(v), \xi \rangle = \frac{1}{2}\omega(\xi v, v)$. The algebra $\mathbb{K}[V^*]$ has a standard star-product called the *Moyal-Weyl* product. Namely, for $f, g \in \mathbb{K}[V^*]$ set $f * g := m(\exp(\frac{\omega}{2}\hbar^2)f \otimes g)$. Here $m : \mathbb{K}[V^*] \otimes \mathbb{K}[V^*] \rightarrow \mathbb{K}[V^*]$ stands for the multiplication map, while $\omega \in \bigwedge^2 V^*$ is assumed to act on $\mathbb{K}[V^*] \otimes \mathbb{K}[V^*]$ via contraction. The quantum algebra $\mathbb{K}[V^*][\hbar]$ is naturally identified with the "homogeneous" version \mathbf{A}_\hbar of the Weyl algebra of V , $\mathbf{A}_\hbar := T(V)[\hbar]/(u \otimes v - v \otimes u - \hbar^2 \omega(u, v), u, v \in V)$.

Our second example is more involved although is also standard.

Let G be a connected reductive algebraic group. The cotangent bundle $X := T^*G$ of G is equipped with a canonical symplectic form ω . Set $\tilde{G} := G \times G$ and consider the \tilde{G} -action on X induced from the two-sided action of \tilde{G} on G . In more detail, we can identify T^*G with

$G \times \mathfrak{g}^*$ using the trivialization by left-invariant forms. Then the "left" action of G on X is given by $g.(g_1, \alpha) = (gg_1, \alpha)$, while the "right" action is $g.(g_1, \alpha) = (g_1g^{-1}, g.\alpha)$. Finally, let \mathbb{K}^\times act on X by $t.(g_1, \alpha) = (g_1, t^{-2}\alpha)$. Clearly, ω is \tilde{G} -invariant and $t.\omega = t^2\omega$. A moment map $\mu : X \rightarrow \tilde{\mathfrak{g}}^* = \mathfrak{g}^* \oplus \mathfrak{g}^*$ is given by $(g, \alpha) \mapsto (g.\alpha, \alpha)$.

Pick a $\tilde{G} \times \mathbb{K}^\times$ -invariant connection ∇ on X and produce the star-product $*$ from ∇ . From the grading considerations, we see that $\mathbb{K}[X][\hbar]$ is a subalgebra in the quantum algebra $\mathbb{K}[X][[\hbar]]$.

There is a standard alternative description of $\mathbb{K}[X][\hbar]$, see, for example, Subsection 7.1 of [40]. Consider the algebra $\mathcal{D}(G)$ of linear differential operators on G . Let $F_i \mathcal{D}(G)$ be the space of differential operators of order $\leq i/2$. Consider the Rees algebra $\mathcal{D}_\hbar(G) := \bigoplus_{i=0}^\infty F_i \mathcal{D}(G) \hbar^i \subset \mathcal{D}(G)[\hbar]$ of $\mathcal{D}(G)$. Then there is a $\tilde{G} \times \mathbb{K}^\times$ -equivariant isomorphism $\mathbb{K}[X][\hbar] \cong \mathcal{D}_\hbar(G)$ of $\mathbb{K}[\hbar]$ -algebras.

Taking the G -invariants in the algebra $\mathbb{K}[T^*G][\hbar] \cong \mathcal{D}_\hbar(G)$ (say for the left G -action), we get a new (star-)product on $\mathbb{K}[\mathfrak{g}^*][\hbar] = \mathbb{K}[T^*G][\hbar]^G$. But $\mathcal{D}_\hbar(G)^G$ is nothing else but a homogeneous version \mathcal{U}_\hbar of the universal enveloping algebra \mathcal{U} of \mathfrak{g} , $\mathcal{U}_\hbar := T(\mathfrak{g})[\hbar]/(\xi \otimes \eta - \eta \otimes \xi - \hbar^2[\xi, \eta], \xi, \eta \in \mathfrak{g})$. In the next subsection we will use a similar recipe to define a W-algebra.

3.2. Equivariant Slodowy slices and W-algebras. A variety we need in the approach to W-algebras from [37] is as follows. Recall the Slodowy slice $S \subset \mathfrak{g}^*$, Subsection 2.4. Set $X := G \times S \subset G \times \mathfrak{g}^* = T^*G$. The variety X is called the *equivariant Slodowy slice*. Clearly, $X \subset T^*G$ is stable with respect to the left G -action. Also it is stable under the restriction of the right G -action to $Q = Z_G(e, \hbar, f)$. Finally, X is stable under a *Kazhdan* \mathbb{K}^\times -action given by $t.(g, \alpha) = (g\gamma(t)^{-1}, t^{-2}\gamma(t)\alpha)$, where $\gamma : \mathbb{K}^\times \rightarrow G$ was introduced in Subsection 2.4. Consider the 2-form ω on X obtained by the restriction of the natural symplectic form from T^*G . One can show that ω is non-degenerate. So X becomes a symplectic variety. It satisfies the assumptions in the beginning of the previous subsection with $\tilde{G} := G \times Q$, the Kazhdan action of \mathbb{K}^\times and a moment map $X \rightarrow \mathfrak{g}^* \oplus \mathfrak{q}^*$ restricted from T^*G .

Pick a $\tilde{G} \times \mathbb{K}^\times$ -invariant symplectic connection ∇ on X and produce a star-product $f * g = \sum_{i=0}^\infty D_i(f, g) \hbar^{2i}$, using the Fedosov construction. [37], Proposition 2.1.5 implies that $\mathbb{K}[X][\hbar] \subset \mathbb{K}[X][[\hbar]]$ is closed with respect to the star-product. We call the quantum algebra $\mathbb{K}[X][\hbar]$ a *homogeneous equivariant W-algebra* and denote it by $\widetilde{\mathcal{W}}_\hbar$. A *homogeneous W-algebra* is, by definition, $\mathcal{W}_\hbar := \widetilde{\mathcal{W}}_\hbar^G$. Finally, define a W-algebra \mathcal{W} as $\mathcal{W}_\hbar/(\hbar - 1)$. So, as a vector space \mathcal{W} is the same as $\mathbb{K}[S]$ but the product on \mathcal{W} is given by $fg := \sum_{i=0}^\infty D_i(f, g)$.

The algebra \mathcal{W} comes equipped with

- a filtration $F_i \mathcal{W}$ induced from the grading on \mathcal{W}_\hbar .
- an action of Q .
- a homomorphism (in fact an embedding) $\mathfrak{q} \hookrightarrow \mathcal{W}$ of Lie algebras such that the adjoint action of \mathfrak{q} on \mathcal{W} coincides with the differential of the Q -action.
- a homomorphism $\mathcal{Z} \rightarrow \mathcal{W}$ (induced from the quantum comoment map $\mathfrak{g} \rightarrow \widetilde{\mathcal{W}}_\hbar$).

It turns out that \mathcal{W} is isomorphic to $U(\mathfrak{g}, e)$. More precisely, we have the following result.

Theorem 3.2 ([37], Corollary 3.3.3). *There is a filtration preserving isomorphism $\mathcal{W} \rightarrow U(\mathfrak{g}, e)$.*

One can prove, in addition, that this isomorphism is Q -equivariant (although this is not written down explicitly) and intertwines the homomorphisms $\mathcal{Z} \rightarrow \mathcal{W}, U(\mathfrak{g}, e)$ (this is proved in [38], the end of Subsection 2.2).

3.3. Decomposition theorem. Let x denote the point $(1, \chi) \in X \subset T^*G = G \times \mathfrak{g}^*$. We remark that the orbit Gx is closed (as any orbit in T^*G) and also $Q \times \mathbb{K}^\times$ -stable. Consider the formal neighborhoods $(T^*G)_{Gx}^\wedge, X_{Gx}^\wedge$ of Gx in T^*G and X and the formal neighborhood $(V^*)_0^\wedge$ of 0 in V^* . Being defined by bidifferential operators, the star-products on $\mathbb{K}[T^*G][\hbar], \mathbb{K}[X][\hbar], \mathbb{K}[V^*][\hbar]$ extend to the corresponding completions $\mathbb{K}[T^*G]_{Gx}^\wedge[[\hbar]], \mathbb{K}[X]_{Gx}^\wedge[[\hbar]], \mathbf{A}_h^\wedge := \mathbb{K}[V^*]_0^\wedge[[\hbar]]$.

Taking the G -invariants in $\mathbb{K}[T^*G]_{Gx}^\wedge[[\hbar]], \mathbb{K}[X]_{Gx}^\wedge[[\hbar]]$ we get star-products on the completions $\mathcal{U}_h^\wedge := \mathbb{K}[\mathfrak{g}^*]_\chi^\wedge, \mathcal{W}_h^\wedge := \mathbb{K}[S]_\chi^\wedge[[\hbar]]$. The algebras $\mathcal{U}_h^\wedge, \mathcal{W}_h^\wedge, \mathbf{A}_h^\wedge$ come equipped with natural (complete and separated) topologies. We remark that the completions $\mathcal{U}_h^\wedge, \mathcal{W}_h^\wedge, \mathbf{A}_h^\wedge$ can be defined completely algebraically, as the inverse limits of $\mathcal{U}_h, \mathcal{W}_h, \mathbf{A}_h$ with respect to the powers of appropriate maximal ideals, see [38], Subsection 2.4 for details.

The following theorem follows from [37], Theorem 3.3.1.

Theorem 3.3. *There is a $Q \times \mathbb{K}^\times$ -equivariant isomorphism $\Phi_h : \mathcal{U}_h^\wedge \rightarrow \mathbf{A}_h^\wedge \widehat{\otimes}_{\mathbb{K}[[\hbar]]} \mathcal{W}_h^\wedge$ of topological $\mathbb{K}[[\hbar]]$ -algebras.*

Here $\widehat{\otimes}$ stands for the completed tensor product: we take the usual tensor product of topological $\mathbb{K}[[\hbar]]$ -algebras and then complete it with respect to the induced topology.

Theorem 3.3 is extremely important in the study of W -algebras. It can be used to prove Theorem 3.2, to prove the category equivalence theorems 4.1, 4.3 in the next section, and also to relate the sets of two-sided ideals of \mathcal{U} and of \mathcal{W} , see Section 5.

4. CATEGORY EQUIVALENCES

This section is devoted to the description of two category equivalences between suitable categories of \mathcal{W} -modules and of \mathcal{U} -modules. In the first subsection we recall an equivalence proved by Skryabin in [57]. This is an equivalence between the category of all \mathcal{W} -modules and the category of *Whittaker* \mathcal{U} -modules. Then we discuss some corollaries of Skryabin's theorem, in particular, a localization theorem due to Ginzburg, [26]. Subsection 4.2 deals with a ramification of Skryabin's equivalence conjectured in [10] and proved in [39]. This is an equivalence between the *category* \mathcal{O} for a W -algebra and the category of *generalized Whittaker* \mathcal{U} -modules.

4.1. Whittaker modules and Skryabin's equivalence. Recall that in Subsection 2.3 we have defined the W -algebra $\mathcal{W} = U(\mathfrak{g}, e)$ as the quantum Hamiltonian reduction $(\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi)^{\text{ad } \mathfrak{m}}$. In other words, $\mathcal{W} = \text{End}_{\mathcal{U}}(\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi)^{\text{op}}$. In particular, $\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi$ is a \mathcal{U} - \mathcal{W} -bimodule.

We say that a \mathcal{U} -module M is *Whittaker* if the action of \mathfrak{m}_χ on M is locally nilpotent. For instance, $\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi$ is easily seen to be Whittaker. Whittaker modules form a Serre subcategory in the category $\mathcal{U}\text{-Mod}$. Denote the category of Whittaker \mathcal{U} -modules by Wh .

The bimodule $\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi$ gives rise to the following functors:

$$\begin{aligned} \text{Wh} \rightarrow \mathcal{W}\text{-Mod} : M &\mapsto \text{Hom}_{\mathcal{U}}(\mathcal{U}/\mathcal{U}\mathfrak{m}_\chi, M) = M^{\mathfrak{m}_\chi} := \\ &:= \{m \in M : \xi m = \langle \chi, \xi \rangle m, \forall \xi \in \mathfrak{m}\}. \\ \mathcal{W}\text{-Mod} \rightarrow \text{Wh} : N &\mapsto \mathcal{U}/\mathcal{U}\mathfrak{m}_\chi \otimes_{\mathcal{W}} N. \end{aligned}$$

We denote the second functor by Sk .

The following important theorem was proved in [57].

Theorem 4.1. *The functors above are quasi-inverse equivalences.*

Let us mention several important corollaries of this theorem.

The Beilinson-Bernstein localization theorem, [4], is a crucial result in the representation theory of \mathcal{U} . There is an analog of this theorem for W-algebras due to Ginzburg, [26]. See also [18] for an alternative approach.

Recall the Beilinson-Bernstein theorem. Pick a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let $\Delta \subset \mathfrak{h}^*$ be the root system, W the Weyl group, and $\Pi \subset \Delta$ be a system of simple roots. Recall the dot action of W on \mathfrak{h}^* given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, where, as usual, ρ stands for the half of the sum of all positive roots. The center \mathcal{Z} of \mathcal{U} gets identified via the Harish-Chandra isomorphism with the invariant algebra $\mathbb{K}[\mathfrak{h}^*]^W$, the invariants are taken with respect to the dot action.

To any $\lambda \in \mathfrak{h}^*$ one assigns a sheaf \mathcal{D}_λ of twisted differential operators on the flag variety \mathcal{B} of G , see [4]. The algebra $\Gamma(\mathcal{B}, \mathcal{D}_\lambda)$ of global sections is naturally identified with the quotient $\mathcal{U}_\lambda := \mathcal{U}/\mathcal{U}I_\lambda$, where I_λ denotes the maximal ideal of $W \cdot \lambda$ in \mathcal{Z} . So to a \mathcal{D}_λ -module M one can assign the \mathcal{U}_λ -module $\Gamma(\mathcal{B}, M)$. The functor $\Gamma(\mathcal{B}, \bullet)$ has a left adjoint: the localization functor $\mathcal{D}_\lambda \otimes_{\Gamma(\mathcal{B}, \mathcal{D}_\lambda)} \bullet$. The Beilinson-Bernstein theorem states that the functor $\Gamma(\mathcal{B}, \bullet)$ is an equivalence provided λ is regular and dominant, i.e., $\langle \lambda + \rho, \alpha \rangle \notin \mathbb{Z}_{\leq 0}$ for any $\alpha \in \Delta$.

Let us explain some details on Ginzburg's localization theorem. For more details the reader is referred to [26].

One can consider the sheaf \mathcal{D}_λ as a quantization of the symplectic variety $T^*\mathcal{B}$. An analog of $T^*\mathcal{B}$ for \mathcal{W} is the *Slodowy variety* defined as follows. The action of G on $T^*\mathcal{B}$ is Hamiltonian, the Springer resolution morphism $\mu : T^*\mathcal{B} \rightarrow \mathfrak{g}^*$ is a moment map. Recall the projection $\pi : \mathfrak{g}^* \rightarrow \mathfrak{m}^*$. Then $\pi \circ \mu$ is a moment map for the M -action on $T^*\mathcal{B}$. By definition, the Slodowy variety \mathcal{S} is the Hamiltonian reduction $(\pi \circ \mu)^{-1}(\chi|_{\mathfrak{m}})/M$.

To define an analog of the sheaf \mathcal{D}_λ in the W-algebra setting Ginzburg uses the language of *directed algebras* (one can also use the language of microlocal sheaves, see [18]). Once this analog is defined the Beilinson-Bernstein theorem transfers to the W-algebra setting verbatim. The scheme of the proof is as follows: one introduces the notion of a Whittaker \mathcal{D}_λ -module, shows that the functors in the Beilinson-Bernstein theorem restrict to equivalences between the Whittaker subcategories, and then uses the Skryabin theorem.

A related development is as follows. Let L be a finite dimensional \mathfrak{g} -module, and M be a Whittaker \mathfrak{g} -module. Then $L \otimes M$ is again a Whittaker \mathfrak{g} -module. This allows to define tensor products of finite dimensional \mathfrak{g} -modules with \mathcal{W} -modules. These tensor products are studied in detail in [27].

4.2. Category \mathcal{O} for W-algebras. In the representation theory of \mathcal{U} a crucial role is played by the category \mathcal{O} established by Bernstein, I. Gelfand and S. Gelfand in [5]. There is an analog of the BGG category \mathcal{O} for \mathcal{W} introduced by Brundan, Goodwin and Kleshchev in [10]. The most important result about this category is that it is equivalent to a certain category of *generalized Whittaker \mathcal{U} -modules*, [39]. Our exposition follows [39].

Recall the group $Q := Z_G(e, h, f)$ acting on \mathcal{W} and an embedding $\mathfrak{q} \hookrightarrow \mathcal{W}$, see Subsections 2.6, 3.2. Pick a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{q}$ and set $\mathfrak{l} := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t})$. Then \mathfrak{l} is a minimal Levi subalgebra in \mathfrak{g} containing e . Further, pick an integral (=lying in the character lattice of the corresponding maximal torus of Q) element $\theta \in \mathfrak{t}$ with $\mathfrak{z}_{\mathfrak{g}}(\theta) = \mathfrak{l}$. A category we are going to consider will depend on θ .

Consider the decomposition $\mathcal{W} = \bigoplus_{i \in \mathbb{Z}} \mathcal{W}_i$, where $\mathcal{W}_i := \{w \in \mathcal{W} | [\theta, w] = iw\}$. Set $\mathcal{W}_{\geq 0} := \bigoplus_{i \geq 0} \mathcal{W}_i$, $\mathcal{W}_{> 0} := \bigoplus_{i > 0} \mathcal{W}_i$, $\mathcal{W}_{\geq 0}^+ := \mathcal{W}_{\geq 0} \cap \mathcal{W}\mathcal{W}_{> 0}$. Then $\mathcal{W}_{\geq 0}$ is a subalgebra in \mathcal{W} , while $\mathcal{W}_{> 0}$ and $\mathcal{W}_{\geq 0}^+$ are two-sided ideals in $\mathcal{W}_{\geq 0}$.

We say that a \mathcal{W} -module N belongs to the category $\mathcal{O}(\theta)$ (in [39] this category was denoted by $\tilde{\mathcal{O}}^t(\theta)$) if

- N is finitely generated.
- $\mathfrak{t} \subset \mathcal{W}$ acts on N by diagonalizable endomorphisms.
- $\mathcal{W}_{>0}$ acts on N by locally nilpotent endomorphisms.

For example, when e is distinguished (i.e., $\mathfrak{q} = \{0\}$), then $\mathcal{O}(\theta)$ consists precisely of all finite dimensional \mathcal{W} -modules. In this case the notion of the category \mathcal{O} is pretty useless. The other extreme is the case when e is principal in \mathfrak{l} . We will see below that in this case we can say a lot about $\mathcal{O}(\theta)$.

Let us present an important construction of a module in $\mathcal{O}(\theta)$. Pick a $\mathcal{W}_{\geq 0}/\mathcal{W}_{\geq 0}^+$ -module N^0 with diagonalizable \mathfrak{t} -action (e.g., irreducible). Define the *Verma* module $\Delta^\theta(N^0)$ by $\Delta^\theta(N^0) := \mathcal{W} \otimes_{\mathcal{W}_{\geq 0}} N^0$.

The properties of $\mathcal{O}(\theta)$ are quite expectable.

Proposition 4.2. (1) *If N^0 is irreducible, then $\Delta^\theta(N^0)$ has a unique irreducible quotient, say $L^\theta(N^0)$.*
 (2) *Any irreducible module in $\mathcal{O}(\theta)$ is isomorphic to $L^\theta(N^0)$ for unique N^0 .*
 (3) *Let $N \in \mathcal{O}(\theta)$ be such that all \mathfrak{t} -eigenspaces in N are finite dimensional. Then N has finite length.*
 (4) *$\Delta^\theta(N^0)$ with $\dim N^0 < \infty$ satisfies the assumptions of (3).*

This is proved in [10], Theorem 4.5, Corollary 4.12 (in [10] a bit different definition was used, in particular, the assumption in (3) was a part of the definition, but this does not matter).

The most crucial property of $\mathcal{O}(\theta)$ is that it is equivalent to a certain category of \mathcal{U} -modules. To define this category we need some more notation.

Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be the decomposition into the eigenspaces of $\text{ad } \theta$. In particular, $\mathfrak{l} = \mathfrak{g}_0$. Form the subalgebra $\underline{\mathfrak{m}} \subset \mathfrak{g}_0$ by analogy with $\mathfrak{m} \subset \mathfrak{g}$ but using the pair (\mathfrak{g}_0, e) instead of (\mathfrak{g}, e) . We define the \mathcal{W} -algebra $\mathcal{W}^0 := U(\mathfrak{g}_0, e)$. This notation is different from [39] but agrees with [40]. Consider the subalgebra $\tilde{\mathfrak{m}} := \underline{\mathfrak{m}} \oplus \mathfrak{g}_{>0} \subset \mathfrak{g}$ (where $\mathfrak{g}_{>0} := \bigoplus_{i>0} \mathfrak{g}_i$) and set $\tilde{\mathfrak{m}}_\chi := \{\xi - \langle \chi, \xi \rangle, \xi \in \tilde{\mathfrak{m}}\}$. The element $\chi \in \mathfrak{g}^*$ is \mathfrak{t} -invariant and so vanishes on $\mathfrak{g}_{>0}$. Hence $\tilde{\mathfrak{m}}_\chi = \underline{\mathfrak{m}}_\chi \times \mathfrak{g}_{>0}$.

We say that a \mathcal{U} -module M is a *generalized Whittaker* module (for e and θ) if

- M is finitely generated.
- \mathfrak{t} acts on M by diagonalizable endomorphisms.
- $\tilde{\mathfrak{m}}_\chi$ acts on M by locally nilpotent endomorphisms.

The category of generalized Whittaker modules will be denoted by $\text{Wh}(\theta)$ (this notation is again different from the one used in [39]).

Again, one can define a Verma module in $\text{Wh}(\theta)$. Let N^0 be a \mathcal{W}^0 -module with diagonalizable \mathfrak{t} -action. Let $\text{Sk}_0 : \mathcal{W}^0\text{-Mod} \rightarrow U(\mathfrak{g}_0)\text{-Mod}$ be the Skryabin functor (for the pair \mathfrak{g}_0, e). Define the Verma module $\Delta^{e,\theta}(N^0) := \mathcal{U} \otimes_{U(\mathfrak{g}_{\geq 0})} \text{Sk}_0(N^0)$, where $U(\mathfrak{g}_{\geq 0})$ acts on $\text{Sk}_0(N^0)$ via a natural epimorphism $U(\mathfrak{g}_{\geq 0}) \rightarrow U(\mathfrak{g}_0)$.

The following theorem is (a part of) the main result of [39].

Theorem 4.3 ([39], Theorem 4.1). *There is an isomorphism $\Psi : \mathcal{W}^0 \rightarrow \mathcal{W}_{\geq 0}/\mathcal{W}_{\geq 0}^+$ and an equivalence $\mathcal{K} : \text{Wh}(\theta) \rightarrow \mathcal{O}(\theta)$ of abelian categories such that the functors $\mathcal{K}(\Delta^{e,\theta}(\bullet))$ and $\Delta^\theta(\Psi_*(\bullet))$ from the category of \mathfrak{t} -diagonalizable \mathcal{W}^0 -modules to $\mathcal{O}(\theta)$ are isomorphic. Here Ψ_* denotes the push-forward functor with respect to the isomorphism Ψ .*

Let us make a remark on an isomorphism Ψ . Such an isomorphism was first established in [10]. It is however not completely clear if one can use the isomorphism from [10] in Theorem 4.3. A peculiar feature of both isomorphisms is that they do not intertwine the embeddings $\mathfrak{t} \hookrightarrow \mathcal{W}_{\geq 0}/\mathcal{W}_{\geq 0}^+, \mathcal{W}^0$ but rather induce a shift on \mathfrak{t} . Since this shift will be of importance later we will give some details, see Remark 5.5 in [39]. Namely, let ι^0, ι denote the embeddings of \mathfrak{t} to $\mathcal{W}^0, \mathcal{W}_{\geq 0}/\mathcal{W}_{\geq 0}^+$, respectively. Then we have $\iota(\xi) = \Psi(\iota^0(\xi)) - \langle \delta, \xi \rangle$ for an element $\delta \in \mathfrak{t}^*$ constructed as follows. Pick a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ containing \mathfrak{t} and h . Let $\Delta_{<0}$ denote the set of all roots α of \mathfrak{g} with $\langle \alpha, \theta \rangle < 0$. Set

$$(4.1) \quad \delta := \sum_{\alpha \in \Delta_{<0}, \langle \alpha, h \rangle = 1} \frac{1}{2} \alpha|_{\mathfrak{t}} + \sum_{\alpha \in \Delta_{<0}, \langle \alpha, h \rangle \geq 2} \alpha|_{\mathfrak{t}}.$$

Till the end of the subsection we consider the category $\text{Wh}(\theta)$ in the special case when e is principal in \mathfrak{l} . Here $\text{Wh}(\theta)$ (with a slightly different definition) was studied before by McDowell, [43], by Milicic and Soergel, [45], and by Backelin, [1].

To proceed we need some more notation. Choose a system Π of simple roots such that θ is dominant. Then $\Pi_0 := \{\alpha \in \Pi : \langle \alpha, \theta \rangle = 0\}$ is a system of simple roots for \mathfrak{l} . Let $\Delta_+, \Delta_{\mathfrak{l}+}$ denote the systems of positive roots for \mathfrak{g} and \mathfrak{l} . For a root α let e_α denote a corresponding weight vector in \mathfrak{g} . Further, let $W_{\mathfrak{l}}$ denote the Weyl group of \mathfrak{l} . We have the dot action of W on \mathfrak{h}^* defined as in the previous subsection.

The W-algebra \mathcal{W}^0 is identified with the center $Z(\mathfrak{l})$ of $U(\mathfrak{l})$. So all irreducible \mathcal{W}^0 -module are 1-dimensional. The set of their isomorphism classes is in one-to-one correspondence with orbits of the dot action of $W_{\mathfrak{l}}$ on \mathfrak{h}^* .

One may assume that $e = \sum_{\alpha \in \Pi_0} e_{-\alpha}$. Then $\tilde{\mathfrak{m}}$ is nothing else but the maximal nilpotent subalgebra \mathfrak{n} of \mathfrak{g} corresponding to Π . Also we have $\langle \chi, e_\alpha \rangle \neq 0$ if and only if $\alpha \in \Pi_0$. So we recover the setting of [1], [43], [45].

For $\lambda \in \mathfrak{h}^*$ let us write $\Delta(\lambda), L(\lambda)$ for the Verma and irreducible modules with highest weight λ in the BGG category \mathcal{O} and $\Delta^{e,\theta}(\lambda), L^{e,\theta}(\lambda)$ for the Verma and irreducible modules in $\text{Wh}(\theta)$ corresponding to $W_{\mathfrak{l}} \cdot \lambda$.

In [45] Milicic and Soergel proved that the (infinitesimal) block of $\text{Wh}(\theta)$ corresponding to a regular integral central character is equivalent to the block of the BGG category \mathcal{O} with certain *singular* integral character that can be recovered from Π_0 . For a generalization of this equivalence to the parabolic setting see [61].

For other blocks in $\text{Wh}(\theta)$ (corresponding to singular/non-integral central characters) the situation is more subtle. But still one can relate the multiplicities in \mathcal{O} and in $\text{Wh}(\theta)$. For $\lambda, \mu \in \mathfrak{h}^*$ let $[\Delta(\lambda) : L(\mu)], [\Delta^{e,\theta}(\lambda) : L^{e,\theta}(\mu)]$ denote the multiplicities in the corresponding categories.

Theorem 4.4 ([1], Theorem 6.2). *Let $\lambda, \mu \in \mathfrak{h}^*$. If*

- (1) $\lambda \in W \cdot \mu$,
- (2) *and there is $w \in W_{\mathfrak{l}}$ such that $w \cdot \mu$ is antidominant for \mathfrak{l} and $\lambda - w \cdot \mu \in \text{Span}_{\mathbb{Z}_{\geq 0}}(\Delta^+)$,*

then $[\Delta^{e,\theta}(\lambda) : L^{e,\theta}(\mu)] = [\Delta(\lambda) : L(w \cdot \mu)]$. Otherwise, $[\Delta^{e,\theta}(\lambda) : L^{e,\theta}(\mu)] = 0$.

An element $\lambda \in \mathfrak{h}^*$ is said to be antidominant for \mathfrak{l} if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{>0}$ for any $\alpha \in \Delta_{\mathfrak{l}+}$.

5. IDEALS IN $U(\mathfrak{g})$ VERSUS IDEALS IN \mathcal{W}

In this section we will construct maps between the sets $\mathfrak{Id}(\mathcal{U})$ and $\mathfrak{Id}(\mathcal{W})$ of two-sided ideals in \mathcal{U} and \mathcal{W} , respectively. This is done in the first two subsections. In Subsection

5.3 we explain how these maps allow to relate (isomorphism classes of) finite dimensional irreducible \mathcal{W} -modules to primitive ideals $\mathcal{J} \subset \mathcal{U}$ such that the associated variety $V(\mathcal{U}/\mathcal{J})$ is $\overline{\mathbb{O}}$. We conclude the section with some remarks in Subsection 5.4.

5.1. **Map $\bullet_{\dagger} : \mathfrak{Id}(\mathcal{U}) \rightarrow \mathfrak{Id}(\mathcal{W})$.** Recall the algebras $\mathcal{U}_h, \mathbf{A}_h$ from Subsection 3.1, \mathcal{W}_h from Subsection 3.2, and the topological algebras $\mathcal{U}_h^{\wedge}, \mathbf{A}_h^{\wedge}, \mathcal{W}_h^{\wedge}$ from Subsection 3.3. By Theorem 3.3, $\mathcal{U}_h^{\wedge} \cong \mathbf{A}_h^{\wedge} \widehat{\otimes}_{\mathbb{K}[[h]]} \mathcal{W}_h^{\wedge}$. Let us introduce suitable sets of ideals of $\mathcal{U}_h, \mathcal{U}_h^{\wedge}, \mathcal{W}_h^{\wedge}, \mathcal{W}_h$. Namely, let $\mathfrak{Id}_h(\mathcal{U}_h)$ denote the set of \mathbb{K}^{\times} -stable \hbar -saturated two-sided ideals in \mathcal{U}_h (an ideal $\mathcal{J}_h \subset \mathcal{U}_h$ is said to be \hbar -saturated if $\hbar a \in \mathcal{J}_h$ implies $a \in \mathcal{J}_h$, in other words, if the quotient $\mathcal{U}_h/\mathcal{J}_h$ is a flat $\mathbb{K}[[h]]$ -module). Define the sets $\mathfrak{Id}_h(\mathcal{U}_h^{\wedge}), \mathfrak{Id}_h(\mathcal{W}_h), \mathfrak{Id}_h(\mathcal{W}_h^{\wedge})$ in a similar way. We define a map \bullet_{\dagger} as the composition

$$(5.1) \quad \mathfrak{Id}(\mathcal{U}) \xrightarrow{(a)} \mathfrak{Id}_h(\mathcal{U}_h) \xrightarrow{(b)} \mathfrak{Id}_h(\mathcal{U}_h^{\wedge}) \xrightarrow{(c)} \mathfrak{Id}_h(\mathcal{W}_h^{\wedge}) \xrightarrow{(d)} \mathfrak{Id}_h(\mathcal{W}_h) \xrightarrow{(e)} \mathfrak{Id}(\mathcal{W}).$$

Let us describe the intermediate maps.

(a): this map sends $\mathcal{J} \in \mathfrak{Id}(\mathcal{U})$ to $R_h(\mathcal{J}) := \bigoplus (\mathcal{J} \cap F_i \mathcal{U}) \hbar^i$. It is a bijection, the inverse map sends $\mathcal{J}_h \in \mathfrak{Id}_h(\mathcal{U}_h)$ to its image under the natural epimorphism $\mathcal{U}_h \twoheadrightarrow \mathcal{U}_h/(\hbar - 1) = \mathcal{U}$.

(b): this map sends $\mathcal{J}_h \in \mathfrak{Id}_h(\mathcal{U}_h)$ to its closure $\mathcal{J}_h^{\wedge} \subset \mathcal{U}_h^{\wedge}$. Equivalently, $\mathcal{J}_h^{\wedge} = \mathcal{U}_h^{\wedge} \mathcal{J}_h$. This map is neither injective (but it is easy to say when two ideals have the same image) nor surjective.

(c): this map sends $\mathcal{J}_h' \in \mathfrak{Id}_h(\mathcal{U}_h^{\wedge})$ to $\mathcal{I}_h' := \mathcal{J}_h' \cap \mathcal{W}_h^{\wedge}$. It is a bijection: its inverse sends $\mathcal{I}_h' \in \mathfrak{Id}_h(\mathcal{W}_h^{\wedge})$ to $\mathbf{A}_h^{\wedge} \widehat{\otimes}_{\mathbb{K}[[h]]} \mathcal{I}_h'$.

(d): this map sends $\mathcal{I}_h' \in \mathfrak{Id}_h(\mathcal{W}_h^{\wedge})$ to $\mathcal{I}_h := \mathcal{I}_h' \cap \mathcal{W}_h$. It is again a bijection, its inverse sends $\mathcal{I}_h \in \mathfrak{Id}_h(\mathcal{W}_h)$ to its closure.

(e): this map is analogous to the inverse of (a).

Proposition 5.1. *The map $\mathcal{J} \mapsto \mathcal{J}_{\dagger}$ has the following properties.*

- (1) \mathcal{J}_{\dagger} is Q -stable.
- (2) $\text{gr } \mathcal{W}/\mathcal{J}_{\dagger}$ is the pull-back of the $\mathbb{K}[\mathfrak{g}^*]$ -module $\text{gr } \mathcal{U}/\mathcal{J}$ to $S \subset \mathfrak{g}^*$.
- (3) $\mathcal{J}_{\dagger} = \mathcal{W}$ if and only if $\mathbb{O} \cap V(\mathcal{U}/\mathcal{J}) = \emptyset$.
- (4) \mathcal{J}_{\dagger} is a proper ideal of finite codimension in \mathcal{W} if and only if $\overline{\mathbb{O}}$ is an irreducible component of $V(\mathcal{U}/\mathcal{J})$. In this case $\dim \mathcal{W}/\mathcal{J}_{\dagger}$ equals the multiplicity of \mathcal{U}/\mathcal{J} on \mathbb{O} .
- (5) The natural map $(\mathcal{J}/\mathcal{J}\mathfrak{m}_{\chi})^{\text{ad m}} \rightarrow (\mathcal{U}/\mathcal{U}\mathfrak{m}_{\chi})^{\text{ad m}}$ is injective. Its image coincides with \mathcal{J}_{\dagger} .

(1) follows directly from the construction. (2) is Proposition 3.4.2 in [37]. (3) and (4) follow from (2). (5) follows from Subsection 3.5 in [38].

5.2. **Map $\bullet^{\dagger} : \mathfrak{Id}(\mathcal{W}) \rightarrow \mathfrak{Id}(\mathcal{U})$.** By definition, \bullet^{\dagger} is the composition

$$\mathfrak{Id}(\mathcal{W}) \rightarrow \mathfrak{Id}_h(\mathcal{W}_h) \rightarrow \mathfrak{Id}_h(\mathcal{W}_h^{\wedge}) \rightarrow \mathfrak{Id}_h(\mathcal{U}_h^{\wedge}) \rightarrow \mathfrak{Id}_h(\mathcal{U}_h) \rightarrow \mathfrak{Id}(\mathcal{U}),$$

where all maps except $\mathfrak{Id}_h(\mathcal{U}_h^{\wedge}) \rightarrow \mathfrak{Id}_h(\mathcal{U}_h)$ are the inverses of the corresponding maps in (5.1). The map $\mathfrak{Id}_h(\mathcal{U}_h^{\wedge}) \rightarrow \mathfrak{Id}_h(\mathcal{U}_h)$ sends $\mathcal{J}_h' \in \mathfrak{Id}_h(\mathcal{U}_h^{\wedge})$ to $\mathcal{J}_h' \cap \mathcal{U}_h$.

Let us list some properties of the map $\mathcal{I} \mapsto \mathcal{I}^{\dagger} : \mathfrak{Id}(\mathcal{W}) \rightarrow \mathfrak{Id}(\mathcal{U})$.

Proposition 5.2. (1) *Let N be a \mathcal{W} -module. Then $\text{Ann}_{\mathcal{W}}(N)^{\dagger} = \text{Ann}_{\mathcal{U}}(\text{Sk}(N))$, where Sk denotes the Skryabin functor, see Subsection 4.1.*

(2) *Let N be a \mathcal{W} -module from the category $\mathcal{O}(\theta)$, see Subsection 4.2. Then $\text{Ann}_{\mathcal{W}}(N)^{\dagger} = \text{Ann}_{\mathcal{U}}(\mathcal{K}(N))$, where \mathcal{K} is the functor from Theorem 4.3.*

(3) *Let \mathcal{I} be an ideal of finite codimension in \mathcal{W} . If \mathcal{I} is prime (resp., completely prime, primitive), then so is \mathcal{I}^{\dagger} .*

- (4) $V(\mathcal{U}/\mathcal{I}^\dagger) = \overline{\mathbb{O}}$ if and only if \mathcal{I} is of finite codimension.
- (5) Recall that the center \mathcal{Z} of \mathcal{U} is identified with the center of \mathcal{W} . Under this identification for any $\mathcal{I} \in \mathfrak{Id}(\mathcal{W})$ we have $\mathcal{I} \cap \mathcal{Z} = \mathcal{I}^\dagger \cap \mathcal{Z}$.
- (6) The map $\mathcal{I} \mapsto \mathcal{I}^\dagger$ is Q -invariant.

Recall that an ideal I in an associative unital algebra A is called prime (resp., completely prime) if a or b lies in I whenever $aAb \subset I$ (resp., $ab \in I$). An ideal I is said to be primitive if it is the annihilator of an irreducible A -module.

(1) is assertion (ii) of [37], Theorem 1.2.2. (2) is a part of [39], Theorem 4.1. (3) stems from [37], Theorem 1.2.2. The "if" part of (4) follows from (1) and [51], Theorem 3.1. The "only if" part follows from the inclusion $(\mathcal{I}^\dagger)_\dagger \subset \mathcal{I}$ that is a direct consequence of our constructions. (5) is assertion (iii) of [37], Theorem 1.2.2. (6) follows directly from the construction.

5.3. Classification of finite dimensional irreducible \mathcal{W} -modules. This subsection is perhaps the most important part of the notes. Here we explain known results about the classification of finite dimensional irreducible \mathcal{W} -modules. We have two results here, both are due to the author, [38],[40]. Both relate the set $\text{Irr}_{fin}(\mathcal{W})$ of (isomorphism classes of) finite dimensional irreducible \mathcal{W} -modules to the set $\text{Prim}_{\mathbb{O}}(\mathcal{U})$ consisting of all primitive ideals $\mathcal{J} \subset \mathcal{U}$ with $V(\mathcal{U}/\mathcal{J}) = \overline{\mathbb{O}}$.

The first result was conjectured by Premet (private communication). To state it we notice that the set $\text{Irr}_{fin}(\mathcal{W})$ is canonically identified with the set $\text{Prim}_{fin}(\mathcal{W})$ of maximal (=primitive) ideals of finite codimension in \mathcal{W} (via taking the annihilator). Thanks to assertions (3),(4) of Proposition 5.2, we see that $N \mapsto \text{Ann}_{\mathcal{W}}(N)^\dagger$ is a map $\text{Irr}_{fin}(\mathcal{W}) \rightarrow \text{Prim}_{\mathbb{O}}(\mathcal{U})$. The group Q acts on $\text{Irr}_{fin}(\mathcal{W})$. The connected component Q° of Q acts trivially because the corresponding action of \mathfrak{q} on \mathcal{W} is by inner derivations. So the Q -action on $\text{Irr}_{fin}(\mathcal{W})$ descends to that of the component group $C(e) = Q/Q^\circ$. By assertion (6) of Proposition 5.2, the map $\text{Irr}_{fin}(\mathcal{W}) \rightarrow \text{Prim}_{\mathbb{O}}(\mathcal{U})$ is $C(e)$ -invariant.

Conjecture 5.3 (Premet). The map $N \mapsto \text{Ann}(N)^\dagger : \text{Irr}_{fin}(\mathcal{W}) \rightarrow \text{Prim}_{\mathbb{O}}(\mathcal{U})$ is surjective and any of its fibers is a single $C(e)$ -orbit.

In [52] Premet proved that any $\mathcal{J} \in \text{Prim}_{\mathbb{O}}(\mathcal{U})$ with *rational* central character lies in the image. In full generality the surjectivity part was first proved in [37], Theorem 1.2.2. Later alternative proofs were found in [26],[53]. The description of fibers is more subtle. It was obtained in [38]. It is a corollary of the following theorem.

Theorem 5.4 ([38], Theorem 1.2.2). *Let \mathcal{I} be a Q -stable ideal of finite codimension in \mathcal{W} . Then $(\mathcal{I}^\dagger)_\dagger = \mathcal{I}$.*

The second result is stated in terms of the category $\mathcal{O}(\theta)$. Let $\theta, \mathfrak{l} = \mathfrak{g}_0, \mathcal{W}^0$ have the same meaning as in Subsection 4.2. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{l}$ and a system of simple roots $\Pi \subset \mathfrak{h}^*$ as in the discussion preceding Theorem 4.4.

Let us introduce some more notation. For $\lambda \in \mathfrak{h}^*$ let $L_0(\lambda)$ stand for the irreducible \mathfrak{g}_0 -module with highest weight λ . Set $J(\lambda) := \text{Ann}_{\mathcal{U}}(L(\lambda)), J_0(\lambda) := \text{Ann}_{U(\mathfrak{g}_0)}(L_0(\lambda))$. According to Duflo, [19], any primitive ideal in \mathcal{U} (resp., in $U(\mathfrak{g}_0)$) has the form $J(\lambda)$ (resp., $J_0(\lambda)$) for some (in general, non-unique) $\lambda \in \mathfrak{h}^*$.

Proposition 5.5. [40], Theorem 5.1.1] *Let N_0 be an irreducible finite dimensional \mathcal{W}^0 -module. If $\text{Ann}_{\mathcal{W}^0}(N_0)^\dagger = J_0(\lambda)$ for some λ , then $\text{Ann}_{\mathcal{W}}(L^\theta(N_0))^\dagger = J(\lambda)$. In particular, $L^\theta(N_0)$ is finite dimensional if and only if $V(\mathcal{U}/J(\lambda)) = \overline{\mathbb{O}}$.*

5.4. Remarks. In the representation theory of \mathcal{U} there are many results on the computation of $V(\mathcal{U}/J(\lambda))$ and on the description of $\text{Prim}_{\mathbb{O}}(\mathcal{U})$. They are due to Joseph, Barbasch-Vogan and others, see, for example, [2],[3],[29]. In particular, it is known that $\text{Prim}_{\mathbb{O}}(\mathcal{U})$ is always non-empty.

Next, we remark that the maps between the sets of ideals upgrade to functors between the categories of Harish-Chandra bimodules, see [26],[38]. The study of these functors is supposed to help to obtain the complete description of $\text{Irr}_{fin}(\mathcal{W})$ itself (not just of the quotient $\text{Irr}_{fin}(\mathcal{W})/C(e)$).

6. ONE-DIMENSIONAL \mathcal{W} -MODULES

6.1. Motivation. The following conjecture was made by Premet.

Conjecture 6.1 ([51], Conjecture 3.1). Any \mathcal{W} -algebra has a one-dimensional representation (equivalently, a two-sided ideal of codimension 1).

At the moment when this text is being written the conjecture is known to be true with exception of several cases in type E_8 , where it is still open.

The reason why Conjecture 6.1 is important is that it implies affirmative answers to some old questions in representation theory of universal enveloping algebras:

- (A) the question of Humphreys on the existence of a small non-restricted representation for semisimple Lie algebras in characteristic p .
- (B) the existence of a completely prime primitive ideal with given associated variety (this question traces back, at least, to Dixmier)

The proof that Conjecture 6.1 implies (A) for $p \gg 0$ is obtained in [53], Theorem 1.4.

The claim that Conjecture 6.1 implies (B) follows from Proposition 5.2: if $\mathcal{I} \subset \mathcal{W}$ has codimension 1, then \mathcal{I}^\dagger is completely prime and $V(\mathcal{U}/\mathcal{I}^\dagger) = \overline{\mathbb{O}}$.

In fact, the implication in the previous paragraph was obtained earlier by Moeglin, [46],[47]. She considered primitive ideals in \mathcal{U} admitting a *Whittaker model*. Using the techniques of [25], one can show that a Whittaker model in the sense of Moeglin is precisely the image of a one-dimensional \mathcal{W} -module under the Skryabin equivalence.

Actually, Moeglin obtained a stronger result: that any ideal admitting a Whittaker model gives rise to a unique quantization (in an appropriate sense) of a suitable covering of \mathbb{O} , see [47] for details.

6.2. Classical algebras. It turns out that Conjecture 6.1 holds for all classical simple Lie algebras. This was proved in [37], Theorem 1.2.3, (1). Let us describe the idea of the proof.

We need to show that there is an ideal of codimension 1 in \mathcal{W} . Thanks to Proposition 5.1, this is the case when there is $\mathcal{J} \in \mathfrak{Id}(\mathcal{U})$ such that $\overline{\mathbb{O}}$ is an irreducible component of $V(\mathcal{U}/\mathcal{J})$ and the multiplicity of \mathcal{U}/\mathcal{J} on $\overline{\mathbb{O}}$ is 1 (this implication also was proved by Moeglin using the language of Whittaker models, see [47]).

Let G be one of the classical groups $\text{SL}_n(\mathbb{K})$, $\text{O}_n(\mathbb{K})$, $\text{Sp}_{2n}(\mathbb{K})$ (depending on \mathfrak{g}). We emphasize that for $\mathfrak{g} = \mathfrak{so}_n$ we need a disconnected group. It turns out that there is an ideal \mathcal{J} in \mathcal{U} such that $\text{gr } \mathcal{U}/\mathcal{J} = \mathbb{K}[\overline{Ge}]$, where gr is taken with respect to the filtration on \mathcal{U}/\mathcal{J} induced from the PBW filtration on \mathcal{U} . Such an ideal is obtained by the quantization of the Kraft-Procesi construction of \overline{Ge} via a Hamiltonian reduction of a vector space, see [34],[35]¹.

¹After [37] was already published I learned that the construction of \mathcal{J} used there (and explained above) was discovered before by R. Brylinski,[15].

In type A more can be said. Form the quotient \mathcal{W}^{ab} of \mathcal{W} by the relations $[a, b], a, b \in \mathcal{W}$. The one-dimensional \mathcal{W} -modules are parametrized by points of $\text{Spec}(\mathcal{W}^{ab})$. In [53], Subsection 3.8, Premet proved that for $\mathfrak{g} = \mathfrak{sl}_n$ the algebra \mathcal{W}^{ab} is the polynomial algebra in $d - 1$ variables, where d is the maximal size of a Jordan block of e . Premet's proof is based on the Brundan-Kleshchev presentation of \mathcal{W} , see Subsection 7.1 for details.

6.3. Parabolic induction. It is easy to prove Conjecture 6.1 when e is even. Indeed, as we have seen in Subsection 2.2, the algebra \mathcal{W} for even e can be embedded into $U(\mathfrak{g}(0))$, see Subsection 2.2. Then we can take any 1-dimensional representation of $U(\mathfrak{g}(0))$ and restrict it to \mathcal{W} .

Premet, [53], observed that a similar result holds in a much more general setting. In the theory of nilpotent elements in semisimple Lie algebras there is a construction called the *Lusztig-Spaltenstein induction*. It was introduced in [41], for a review see, for example, [44]. Namely, let $\underline{\mathfrak{g}} \subset \mathfrak{g}$ be a Levi subalgebra and $\underline{\mathbb{O}} \subset \underline{\mathfrak{g}}$ be a nilpotent orbit. The Lusztig-Spaltenstein induction produces a nilpotent orbit $\mathbb{O} \subset \mathfrak{g}$ from the pair $(\underline{\mathfrak{g}}, \underline{\mathbb{O}})$. We say that \mathbb{O} is induced from $(\underline{\mathfrak{g}}, \underline{\mathbb{O}})$. If e is even, then \mathbb{O} is induced from $(\mathfrak{g}(0), \{0\})$. If \mathbb{O} cannot be induced from a nilpotent orbit in proper Levi subalgebra, \mathbb{O} is called *rigid*.

Theorem 6.2 (Premet, [53]). *Let \mathbb{O} be induced from $(\underline{\mathfrak{g}}, \underline{\mathbb{O}})$. If the algebra $\underline{\mathcal{W}} := U(\underline{\mathfrak{g}}, \underline{e})$, where $\underline{e} \in \underline{\mathbb{O}}$, has a one-dimensional representation, then \mathcal{W} does.*

Premet's proof of Theorem 6.2 is based on the reduction to positive characteristic. Another proof, close in spirit to that for even elements, was found by the author in [40]. Namely, under the assumptions of Theorem 6.2, there is an embedding of \mathcal{W} into a certain *completion* of $\underline{\mathcal{W}}$. The latter acts on all finite dimensional $\underline{\mathcal{W}}$ -modules. So a one-dimensional \mathcal{W} -module again can be obtained by restriction.

6.4. 1-dimensional representations via category $\mathcal{O}(\theta)$. In this subsection we will explain how to apply the category $\mathcal{O}(\theta)$ to the study of one-dimensional representations of \mathcal{W} , see [40].

We use the notation from Subsection 4.2. Let us impose the following condition on a nilpotent element e :

(*) the algebra \mathfrak{q} is semisimple.

It turns out that this condition is satisfied for all rigid nilpotent elements. A proof based on the classification of such elements can be found in [40], Subsection 5.2. It would be very interesting to find a conceptual proof.

Let N^0 be a finite dimensional \mathcal{W}^0 -module. We want a criterium for $L^\theta(N^0)$ to be 1-dimensional. Since $N^0 \hookrightarrow L^\theta(N^0)$, of course, $\dim L^\theta(N^0) = 1$ implies $\dim N^0 = 1$.

The following result follows from Theorem 5.2.1 in [40].

Theorem 6.3. *Suppose the condition (*) holds. Let N^0 be a 1-dimensional \mathcal{W}^0 -module. Then the following conditions are equivalent:*

- (1) $\dim L^\theta(N^0) = 1$.
- (2) $\mathfrak{t} \subset \mathcal{W}^0$ acts on N^0 by δ , where δ is given by (4.1).

When e is of principal Levi type (which is true for all but 2 rigid nilpotent elements in exceptional Lie algebras), then any irreducible \mathcal{W}^0 -module is 1-dimensional (recall that \mathcal{W}^0 is just the center of $U(\mathfrak{l})$).

Combining Theorem 6.3, Proposition 5.5, and assertion (4) of Proposition 5.2 one obtains a criterium for an ideal $\mathcal{J} \subset \mathcal{U}$ to have the form \mathcal{I}^\dagger with $\dim \mathcal{W}/\mathcal{I} = 1$. More precisely, we have the following result, [40], Subsection 5.3.

Corollary 6.4. *Suppose \mathfrak{g} satisfies (*). Let $\theta, \mathfrak{h}, \Pi$ be chosen as in the discussion preceding Theorem 4.4. Let \mathbb{O}_0 denote the adjoint orbit of e in \mathfrak{l} .*

- (1) *Let $\lambda \in \mathfrak{h}^*$ satisfy the following four conditions:*
 - (A) *The associated variety of $U(\mathfrak{l})/J_0(\lambda)$ in \mathfrak{g}_0^* is $\overline{\mathbb{O}_0}$.*
 - (B) *$\dim V(\mathcal{U}/J(\lambda)) \leq \dim \mathbb{O}$.*
 - (C) *$\lambda - \delta$ vanishes on \mathfrak{t} .*
 - (D) *$J_0(\lambda)$ corresponds to an ideal of codimension 1 in \mathcal{W}^0 .*
Then $J(\lambda) = \mathcal{I}^\dagger$ for some ideal $\mathcal{I} \subset \mathcal{W}$ of codimension 1.
- (2) *For any ideal $\mathcal{I} \subset \mathcal{W}$ of codimension 1 there is $\lambda \in \mathfrak{h}^*$ satisfying (A)-(D) and such that $J(\lambda) = \mathcal{I}^\dagger$.*

When e is principal in \mathfrak{l} the condition (A) means that λ is antidominant for \mathfrak{l} , while the condition (D) becomes vacuous.

6.5. Exceptional algebras. Let us summarize what is known about Conjecture 6.1 for exceptional Lie algebras. As Premet checked in [51], \mathcal{W} has a one-dimensional module provided e is a minimal nilpotent element (in an arbitrary simple Lie algebra). His approach was based on an analysis of generators and relations for \mathcal{W} that are not very difficult for minimal nilpotents. Recently Goodwin, Röhrle and Ubly, [28], extended Premet's approach to all rigid nilpotents in G_2, F_4, E_6, E_7 and some rigid nilpotents in E_8 . The result is that for all nilpotent elements they considered a one-dimensional \mathcal{W} -module does exist. They used the GAP program to analyze the relations. "Large" nilpotent elements in E_8 remain to complicated computationally. Maybe, one can deduce Conjecture 6.1 for E_8 from Corollary 6.4.

7. TYPE A

This section is devoted to results concerning W-algebras for $\mathfrak{g} = \mathfrak{sl}_N$ (or $\mathfrak{g} = \mathfrak{gl}_N$). In Subsection 7.1 we very briefly sketch a relation between W-algebras and Yangians. In Subsection 7.2 we mention some other results: the higher level Schur-Weyl duality of Brundan and Kleshchev and the Gelfand-Kirillov conjecture for W-algebras proved by Futorny, Molev and Ovsienko.

7.1. W-algebras vs Yangians. In this subsection we will briefly explain a relationship between W-algebras for $\mathfrak{g} = \mathfrak{gl}_N$ and certain infinite dimensional algebras called *shifted Yangians*. A shifted Yangian is a certain generalization of the usual Yangian for \mathfrak{gl}_n . For a comprehensive treatment of Yangians and related algebras the reader is referred to Molev's book [48]. A relation between Yangians and W-algebras was first observed by Ragoucy and Sorba in [56] and then generalized to shifted Yangians by Brundan and Kleshchev, [12].

The Yangian $Y(\mathfrak{gl}_n)$ can be defined as the algebra generated by elements $t_{ij}^{(r)}, i, j = 1, \dots, n, r \in \mathbb{N}$, subject to the relations

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)}.$$

However, the generators $t_{ij}^{(r)}$ are not convenient to establish a relation between the Yangians and W-algebras. In [11] Brundan and Kleshchev found a new presentation of $Y(\mathfrak{gl}_n)$. Generalizing this presentation they introduced shifted Yangians in [12].

A shifted Yangian $Y_n(\sigma)$ depends on a positive integer n and some *shift matrix* σ . By definition, $\sigma = (s_{ij})_{i,j=1}^n$ is a shift matrix if s_{ij} is a nonnegative integer ("shift") with $s_{ij} + s_{jk} = s_{ik}$ whenever $|i - j| + |j - k| = |i - k|$. By definition, the algebra $Y_n(\sigma)$ is given by generators

$$D_i^{(r)}(1 \leq i \leq n, r > 0), E_i^{(r)}(1 \leq i < n, r > s_{i,i+1}), F_i^{(r)}(1 \leq i < n, r > s_{i+1,i})$$

subject to certain explicit relations (see [12], (2.4)-(2.15)). The shifted Yangian coincides with the usual one when $\sigma = 0$. For $l > s_{1,n} + s_{n,1}$ define the quotient (the truncated shifted Yangian of level l) $Y_{n,l}(\sigma)$ of $Y_n(\sigma)$ by the two-sided ideal generated by $D_i^{(r)}, 1 \leq i \leq n, r > p_1 := l - s_{1,n} - s_{n,1}$.

To establish a relationship between shifted Yangians and W -algebras fix a positive integer n , pick a Young diagram $\lambda = (\lambda_1, \dots, \lambda_n), \lambda_1 \geq \dots \geq \lambda_n \geq 0$ (one can also work with more general diagrams called *pyramids*, see [12], §7 for details), and set $l := \lambda_1$. Then to n and λ one can assign the *shift matrix* $\sigma^\lambda = (s_{ij})_{i,j=1}^n$ by setting $s_{ij} := 0$ for $i \geq j$ and $s_{ij} := \lambda_{n+1-j} - \lambda_{n+1-i}$ for $i < j$. In particular, for the Young diagram of shape $n \times l$, we get $\sigma = 0$.

To λ one assigns a nilpotent element $e_\lambda \in \mathfrak{gl}_N$, where $N := \sum_{i=1}^n \lambda_i$, in the usual way (λ_i are the sizes of the Jordan blocks of e_λ).

Theorem 7.1 ([12], Theorem 10.1). $U(\mathfrak{gl}_N, e_\lambda) \cong Y_{n,l}(\sigma^\lambda)$.

In [13] Brundan and Kleshchev used this theorem to study the representation theory of $U(\mathfrak{gl}_N, e_\lambda)$. In particular, they obtained a classification of finite dimensional irreducible $U(\mathfrak{gl}_N, e)$ -modules (which also follows from Proposition 5.5 thanks to Joseph's computation of $V(U(\mathfrak{gl}_N)/J(\lambda))$, see [29]; we remark that any nilpotent element in \mathfrak{gl}_N is of principal Levi type).

There is a generalization of the results explained above in this subsection to other classical Lie algebras first observed by Ragoucy, [55] and worked out in more detail by J. Brown, [7],[8]. Namely, for orthogonal and symplectic algebras there are analogs of $Y(\mathfrak{gl}_n)$ called *twisted Yangians*. Theorem 7.1 generalizes to twisted Yangians. It is interesting that, similarly to $Y(\mathfrak{gl}_n)$ -case, nilpotent elements arising in this generalization again correspond to partitions with all parts equal. It is unclear whether there is a reasonable shifted version of the twisted Yangians that is related to the W -algebras constructed from arbitrary nilpotent elements.

7.2. Other results. W -algebras in type A enjoy some other interesting properties.

For example, in [14] Brundan and Kleshchev obtained a very nice result: a "higher level" generalization of the classical Schur-Weyl duality. Recall that the classical Schur-Weyl duality relates between polynomial representations of $GL_N(\mathbb{K})$ and representations of the symmetric group S_d in d letters. The Brundan-Kleshchev generalization relates modules over the cyclotomic degenerate Hecke algebra $H_d(\lambda)$ corresponding to a partition λ of N (this algebra is a higher level generalization of S_d) and modules over the W -algebra $U(\mathfrak{gl}_N, e_\lambda)$. For details the reader is referred to [14] or to the review [60] by Wang.

Another result we would like to mention is an analog of the Gelfand-Kirillov conjecture for W -algebras proved in [23].

For a Noetherian domain A let $Q(A)$ denote its skew-field of fractions. Gelfand and Kirillov, [24], conjectured that for any finite dimensional algebraic Lie algebra \mathfrak{a} the skew-field $Q(U(\mathfrak{a}))$ is isomorphic to $Q(\mathbf{A}_l(F_d))$, where F_d is a purely transcendental extension of \mathbb{K} of some degree d and $\mathbf{A}_l(F_d)$ stands for the Weyl algebra of a $2l$ -dimensional symplectic

vector space over F_d . In [24] the conjecture was verified for $\mathfrak{g} = \mathfrak{sl}_n^2$. In [23] Futorny, Molev and Ovsienko proved that the straightforward analog of the Gelfand-Kirillov conjecture holds for $U(\mathfrak{gl}_n, e)$ (and for $U(\mathfrak{sl}_n, e)$) for an arbitrary nilpotent element $e \in \mathfrak{sl}_n$.

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²Recently Premet proved in [54] that the Gelfand-Kirillov conjecture does not hold for \mathfrak{a} of type $B_n(n \geq 3)$, $D_n(n \geq 4)$, E_6 , E_7 , E_8 , F_4 .

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